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Bounded Approximation by Analytic Functions

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We consider the following problem:

Let U be a bounded open subset of the complex plane **C**. What conditions on U ensure that given a bounded analytic function f on U we can find a bounded sequence $\{f_n\}$ of functions in $A(U)$ (the algebra of all continuous complex-valued functions on $\bar{U}$ which are analytic on U) such that $f_n \to f$ pointwise on U?

In this paper we summarize the known results on this problem and indicate some new ones. For a survey of some related questions see [10].

We say that $A(U)$ is *pointwise boundedly dense* (p.b.d.) in $H^\infty(U)$ (the algebra of all bounded analytic functions on U) if a sequence of the type described above can be found for every $f \in H^\infty(U)$. If we can choose the sequence so that, in addition, $\|f_n\| \leqslant \|f\|$ (supremum norms) for each n, we say that $A(U)$ is *strongly pointwise boundedly dense* (s.p.b.d.) in $H^\infty(U)$. In [11] Rubel and Shields, extending earlier work of Farrell [4], proved that $A(U)$ is s.p.b.d. provided U is the interior of a compact set with connected complement (in this case one can take f_n to be a polynomial). Ahern and Sarason [1] proved a slightly more general result by functional-analytic methods.

In [6] Gamelin and Garnett modified the techniques of Vituskin [13], developed for uniform approximation, to attack the present problem. Their results are expressed in terms of the concepts of analytic capacity $\gamma(E)$ and continuous analytic capacity $\alpha(E)$ of the set E, defined as follows:

$$\gamma(E) = \sup\{|f'(\infty)|; f \text{ is analytic outside a compact subset of } E, \\ |f| \text{ is bounded by } 1, f(\infty) = 0\}.$$

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$\alpha(E)$ is defined similarly, except that f is required to be continuous on **C**. For properties of γ, α, see [5, Chapter 8].

In Theorem 2.2 of [6] Gamelin and Garnett state that $A(U)$ is p.b.d. if and only if there exist $r > 1$, $c > 0$ such that

$$\gamma(\Delta(z, \delta) \cap bU) \leqslant c\alpha(\Delta(z, r\delta)\backslash U) \qquad (*)$$

for all $\delta > 0$ and $z \in bU$. [Here $\Delta(z, \delta)$ denotes the open disc with center z and radius δ; bU denotes the boundary of U.]

To extract the most from this result it seems necessary to combine it with functional-analytic techniques. In [2], methods due to Ahern and Sarason [1] and Gamelin [5, Chapter 8] are applied to show that if $A(U)$ is p.b.d. then it is s.p.b.d. Together with the above result of Gamelin and Garnett this can be used to show that we need only assume $(*)$ holds for each $z \in bU$, with c and r depending on z. This has the geometric corollary that if U is the interior of a compact set K such that the inner boundary of K (the set of points of bK which are not boundary points of any component of $\mathbf{C}\backslash K$) is empty, then $A(U)$ is s.p.b.d. [in fact, the set of rational functions with poles off K is s.p.b.d. since, in this case, they are uniformly dense in $A(U)$]. See [13, Chapter 2, Section 5].

By further refining these techniques, Gamelin and Garnett showed in [8] that it suffices to have $(*)$ for each $z \in bU\backslash\bigcup_{n=1}^{\infty} E_n$, where, for each n, E_n is a set of zero length lying on a C^2 arc. Again we have the geometric corollary that if the inner boundary of K lies on such a set $\bigcup_{n=1}^{\infty} E_n$, then the set of rational functions with poles off K is s.p.b.d. in $H^\infty(K^0)$, K^0 being the interior of K.

By a similar argument one can prove:

Let E be the set of points of bU at which $(*)$ holds. Let $f \in H^\infty(U)$, let K be a compact subset of $\overline{U}\backslash E$, and let $\epsilon > 0$. Then we can find $g \in H^\infty(U)$, extending continuously to a neighborhood of E, with $\| g \| \leqslant \| f \|$ and $| g - f | < \epsilon$ on K.

An interesting special case occurs when U has a connected complement. Let $U_1, U_2, \ldots$ be the components of U; each U_i is simply connected. Let $\Delta_1, \Delta_2, \ldots$ be open discs with disjoint closures, and let $\varphi: \bigcup_i \Delta_i \to \bigcup_i U_i$ map Δ_i conformally onto U_i for each i. The radial boundary values φ^* are defined almost everywhere on $\bigcup_i \Gamma_i$, where Γ_i is the boundary of U_i. Then the following are equivalent:

(1) $A(U)$ is s.p.b.d. in $H^\infty(U)$.

(2) $A(U)$ is a dirichlet algebra on bU.

(3) There is a subset E of zero length of $\bigcup_i \Gamma_i$ such that φ^* is defined and (1–1) on $\bigcup_i \Gamma_i\backslash E$.

(2) means that every continuous real function on bU is a uniform limit of real parts of functions in $A(U)$.

(2) $\Rightarrow$ (1) follows from a general result on Dirichlet algebras due to Hoffman and Wermer; see [14].

(1) $\Rightarrow$ (3) is elementary. (1) $\Rightarrow$ (2) was proved by Gamelin and Garnett [8], and a short proof was found by Øksendal [9]. A proof of (3) $\Rightarrow$ (1) is given in [3].

In this connection we mention the following result.

Suppose K is a compact set such that $\mathbf{C}\backslash K^0$ is connected and the inner boundary of K has zero 1/2-dimensional Hausdorff measure. Then $R(K)$, the set of rational functions with poles off K is Dirichlet on bK, and is s.p.b.d. in $H^\infty(K^0)$.

Finally we mention a problem involving uniform approximation: suppose $f \in H^\infty(U)$, and f extends continuously to a subset E of bU. Let $\epsilon > 0$. Can we find $g \in H^\infty(U)$, extending continuously to a neighborhood of E in bU, with $\|f - g\| < \epsilon$? This was shown to be the case when U is a disc by Stray [12], and Gamelin and Garnett [7]. (In fact these authors obtained approximating functions that are analytic in certain neighborhoods of E.) More generally the answer is positive provided the capacity estimate (*) holds at each point of E. For arbitrary U and compact E it turns out that the problem is equivalent to the following one, involving semi-additivity of α:

Does there exist an absolute constant $M > 0$ such that

$$\alpha(K \cup L) \leqslant M(\alpha(K) + \alpha(L))$$

whenever K, L are sets of which one is compact?

See [13] for information about the semi-additivity problem.

References

1. P. Ahern and D. Sarason, On some hypodirichlet algebras of analytic functions, *Amer. J. Math.* **89** (1967), 932–941.
2. A. M. Davie, Bounded approximation and Dirichlet sets *J. Functional Analysis* **6** (1970), 460–467.
3. A. M. Davie, Dirichlet algebras of Analytic functions, *J. Functional Analysis* **6** (1970), 348–356.
4. O. J. Farrell, On approximation by polynomials to a function analytic in a simply connected region, *Bull. Amer. Math. Soc.* **41** (1935), 707–711.
5. T. W. Gamelin, "Uniform Algebras," Prentice-Hall, Englewood Cliffs, NJ, 1969.
6. T. W. Gamelin and J. Garnett, Constructive techniques in rational approximation, *Trans. Amer. Math. Soc.* **143** (1969), 187–200.
7. T. W. Gamelin and J. Garnett, Uniform approximation to bounded analytic functions, *Revista de la Union Matematica Argentina* **25** (1970), 87–94.

8. T. W. Gamelin and J. Garnett, Pointwise bounded approximation and Dirichlet algebras, *J. Functional Analysis* **143** (1969), 187–200.
9. B. Øksendal, $R(X)$ as a Dirichlet algebra and representation of orthogonal measures by differentials, *Math. Scand.* **29** (1971), 87–103.
10. L. A. Rubel, Bounded convergence of analytic functions, *Bull. Amer. Math. Soc.* **77** (1971), 13–24.
11. L. A. Rubel and A. L. Shields, Bounded approximation by polynomials, *Acta Math.* **112** (1964), 145–162.
12. A. Stray, An approximation theorem for subalgebras of H^∞, *Pacific J. Math.* **35** (1970), 511–515.
13. A. G. Vitushkin, Analytic capacity of sets and problems in approximation theory, *Russian Math. Surveys* **22** (1967), 139–200.
14. J. Wermer, "Seminar über Funktionen Algebren," Lecture Notes in Math., No. 1, Springer-Verlag, Berlin, 1964.